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On the number of chord diagrams

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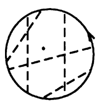
Abstract

We present some combinatorial results in counting various kinds of s.c. chord diagrams. Latter are basic objects in the theory of Vassiliev invariants for knots.

1. Short introduction

A chord diagram (a CD) is an object like this,



i.e. an oriented circle with finitely many dashed chords in it and considered up to isotopy. The mathematical interest of such a object remains the fact, that it is an important part of the combinatorial structure underlying a big class of topological knot invariants, introduced by Vassiliev [20] and for understanding them it is often helpful to consider some pictures like the one above. See e.g. [1] and loc. cit. for a good description of this topic.

In this paper we treat some enumeration problems of certain kinds of CDs connected to the algebraic structures coming from knot theory. The essential difficulty of this enumeration is to determine the number of their linearized relatives, called LCDs, fixed by a certain cyclic permutation of the basepoints. This we achieve by introducing some new objects called GLCDs.

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It should be mentioned, that similar enumeration problems have been treated in another way in several other papers, e.g. [6,8,9,11,14,17].

1.1. Notations

For two numbers $m, n \in \mathbb{N}$ their g.c.d. is denoted (m, n) and $m\%n$ is $m \bmod n$.

If P is a finite set, by the notation $\#P$ we will denote its cardinality and by $\mathscr{P}(P)$ its power set (set of all subsets).

In the following, we will need some arithmetic functions. $\varphi(n)$ will denote the Euler function. A well-known property of these values is that for all $n \in \mathbb{N}_+$

$$\sum_{d|n} \varphi(d) = n. \tag{1}$$

Let

$$(n)_d = \frac{n!}{(n-d)!}$$

denote the number of d-fold ordered choices out of n elements.

The bifactorial $n!!$ of an integral number n is defined by

$$n!! = \prod_{i=0}^{\lfloor (n-1)/2 \rfloor} n - 2i$$

for $n > 0$ and by convention we set $0!! = 1$, $(-1)!! = 1$ and $n!! = 0$ for $n \leqslant -2$.

$[P(x)]_d$ will denote the coefficient of x^d in the polynomial (or power series) P in the formal variable x.

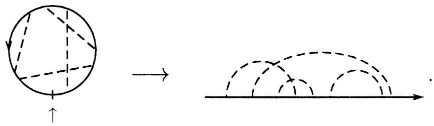
By $\lfloor n \rfloor$ we will mean the greatest integer not greater than n.

The definitions of the various properties of all diagrams in the following, unless not noted in this paper, are classical in the theory of Vassiliev invariants and can be found in various papers on that subject, e.g. [1,2,5,7,19]. (See [3] for an extensive survey on publications on Vassiliev invariants.) We shall just make one general restriction: we will not consider graphs with trivalent vertices of dashed lines, that is, we do not consider Chinese character diagrams in the terminology of [1]. (In fact, knot-theoretically, Chinese character diagrams give an alternative representation of the algebra of chord diagrams, basically because using the various 4T relations within the different diagram spaces [19], trivalent vertices can be resolved.)

2. Counting all, degenerate and symmetric CDs

2.1. Linearized CDs

One can obtain a linearized CD (an LCD) from an usual CD by 'cutting' somewhere the solid line. Then one has something like this



Of course this cutting is not unique (but knot-theoretically it is the same modulo the diagram algebra relations). Both CDs and LCDs are graded by the number of their chords, so the picture above is of degree 4.

Let us use the following notations:

$$C_D = \{\text{CDs of deg } D\}, \quad \sigma_D = \#C_D,$$

$$L_D = \{\text{LCDs of deg } D\}, \quad \lambda_D = \#L_D.$$

A generalization of the LCDs with more than one solid line are the s.c. *string link* (sl-) diagrams (for pictures look e.g. in [2]). Formally, a string link diagram consists of a collection of unclosed solid lines and a collection of dashed lines (chords) such that all ends of all chords lie on mutually distinct points of some of the solid lines, not necessarily equal for the endpoints of the same chord. Let

$$L_{D,k} = \{\text{sl-diagrams with } k \text{ strands of deg } D\}, \quad \lambda_{D,k} = \#L_{D,k}.$$

The motivation to start these considerations was for me the fact, that the number λ_D of LCDs of deg D can be computed very easily. In fact, it is well known to show the following:

Lemma 2.1.

$$\lambda_D = (2D - 1)!!.$$

As a generalization of this fact, one can prove the following statement about $\lambda_{D,k}$:

Lemma 2.2.

$$\lambda_{D,k} = \binom{2D + k - 1}{2D} (2D - 1)!!$$

Proof (sketch). Glue all strands into one and place a mark on the point of each gluing. □

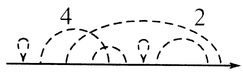
S_{2D} acts on L_D by permuting the order of the base points of the D chords, and in this sense C_D is isomorphic to the orbit space of the cyclic subgroup $\mathbb{Z}_{2D} \subset S_{2D}$ generated by the cycle $z_D = (1\,2\,3\,\ldots 2D)$ on L_D. So, we shall consider the behaviour of LCDs under this action.

Let for $\sigma \in S_{2D}$

$$R_\sigma = \{\text{LCDs } Y \text{ of deg } D \text{ with } \sigma(Y) = Y\}, \quad r_\sigma = \#R_\sigma.$$

2.2. Cyclic CDs and GLCDs

Definition 2.1. A generalized linearized CD (GLCD) is a pair of the following form:

( , c)

where $c \in \mathbb{N}_+$ and the first component is something like a LCD, but has the following two additional features

- If c is even, it may contain self-loops , i.e. chords starting and ending onto the same basepoint.
- Each real chord (a chord which is not a self-loop) is equipped with a number between 0 (in this case we drop the number for convenience) and $c-1$. We will say that it's *coloured* or *labeled* by this number.

Let the GLCDs be graded by the number of the basepoints (not chords !) and the cyclicity of a GLCD be its second component. So the LCDs are exactly GLCDs with cyclicity 1. Then the above picture has degree 10 and cyclicity c with $c \geqslant 5$.

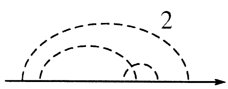
It will be sometimes convenient to drop the cyclicity and take only the first part (what is meant will be clear from the context).

Let

$$\Gamma_{d,c} = \{\text{GLCDs of deg } d \text{ and cycl. } c\} \quad \text{and} \quad \gamma_{d,c} = \#\Gamma_{d,c}.$$

Furthermore, introduce an action of $\mathbb{Z}_d$ on $\Gamma_{d,c}$ in the following manner. We will say how $1 \in \mathbb{Z}_d$ should act on a GLCD.

- It flips self-loops and real chord ends from the right-most position to the left-most
- Each time it flips one of the ends of a real chord, its number changes from k to $n-1-k$, e.g.

( , 7) $\longleftarrow$ ( , 7)

It will turn out as useful to know the cardinality $\gamma_{d,c}$ of $\Gamma_{d,c}$. This is an easy combinatorial task.

If c is odd, then self-loops do not exist and the only non-vanishing case is if d is even. Then we are left with counting LCDs with numbered chords.

If c is even then sum over all possible numbers of  and over the choices to put them between the real chords. Then

$$\gamma_{d,c} = \begin{cases} 0, & 2\nmid c, d, \\ c^{d/2}(d-1)!!, & 2\nmid c, 2|d, \\ \displaystyle\sum_{i=0}^{\lfloor d/2 \rfloor} \binom{d}{2i} \cdot c^i \cdot (2i-1)!! \; 2|c. \end{cases} \tag{2}$$

Let

$$C_{D,c} = \{\text{CDs } Y \text{ of deg } D \text{ with } \mathbb{Z}_c \subset \mathrm{stab}_{\mathbb{Z}_{2D}}(Y) \subset \mathbb{Z}_{2D}\}.$$

In other words

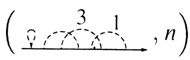
$$C_{D,c} = R_{Z_D^c}/\mathbb{Z}_{2D/c}.$$

A relation between GLCDs and CDs is given by the following:

Theorem 2.1. *There is a bijection*

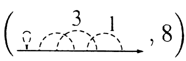
$$C_{D,c} \overset{\Phi_{D,c}}{\leftrightarrow} \Gamma_{2D/c,c}/\mathbb{Z}_{2D/c}.$$

Proof. We should best use an example to demonstrate what we are going to do. Given a GLCD, say



construct the corresponding CD in $C_{7/2n,n}$ as follows:

1. Separate an oriented circle into n arcs and mark on each arc seven basepoints. Number the arcs from 0 to $n-1$ and the basepoints on each arc from 1 to 7.
2. For each real chord in our GLCD and $0 \leqslant m \leqslant n-1$ connect the left end basepoint on the arc m with the right end basepoint on the arc $(m+\mu) \bmod n$, where μ is the color of the chord, e.g. for $n=8$ in the case of arc 0 and the chord colored by 3 in the example we use we get the right chord in Fig. 1.
3. For a self-loop, connect the basepoint in arc m with the same basepoint in arc $(m+n/2) \bmod n$ for m from 0 to $n-1$, e.g. for arc 0 we get the left chord in Fig. 1. (This also explains why chords coming from self-loops do not occur when c is odd and so self-loops do not exist in this case.)
4. Glue all the arcs together and remove all markings on them. The CD in $C_{28,8}$ we obtain from



is shown in Fig. 2.

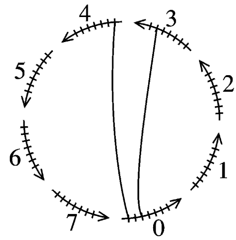


Fig. 1.

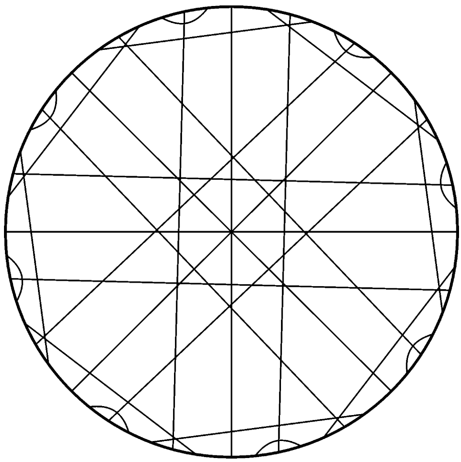


Fig. 2.

Now it is easy to see how to construct the inverse of Φ. To do this, separate for a cyclic CD in $C_{D,c}$ the circle into c pieces with $2D/c$ basepoints and assign the unique numbers to the chords in our GLCD, counting the difference between the arc numbers. If chords in our CD start and end on the same position in different arcs (i.e. we obtain a self-loop), then c must be even and the arcs opposite in order the CD to be cyclic. Now check that the action of $\mathbb{Z}_{2D/c}$ factors out exactly the arbitrarity how to choose the splitting of the baseline into arcs. $\square$

2.3. Counting all CDs

Using Theorem 2.1 we see that LCDs invariant under $d \in \mathbb{Z}_{2D}$ bijectively correspond to GLCDs with cyclicity $c = 2D/d$. Noticing that a LCD is invariant under $d \in \mathbb{Z}_{2D}$ exactly if it is under $d \cdot l$ where $(l, 2D/d) = 1$, we have

$$r_{z_D^c} = \gamma_{(2D,c),2D/(2D,c)}$$

and by Burnside's lemma [10, Lemma 14.3, p. 1058] we get the following combinatorial expression for σ_D.

Theorem 2.2. *With* (2) *one has*

$$\sigma_D = \frac{1}{2D} \sum_{d \cdot c = 2D} \varphi(c) \gamma_{d,c}. \tag{3}$$

This formula is probably originally due to Jean Bétréma [16].

2.4. Symmetric CDs

A variation of the enumeration problem is to count CDs up to mirror images (or equivalently, up to change of orientation of the solid line). Let

$$\hat{\sigma}_D = \#\{\text{CDs of degree } D\}/\text{symmetry}$$

and

$$\sigma_D^{\text{sym}} = \#\{\text{symm. CDs s of degree } D\}.$$

Then clearly,

$$\hat{\sigma}_D = \frac{\sigma_D + \sigma_D^{\text{sym}}}{2}. \tag{4}$$

$\hat{\sigma}_D$ can also be computed using Burnside's lemma. In view of (4) it is more convenient to give $\hat{\sigma}_D$ in terms of σ_D^{sym}, since the resulting formula for σ_D^{sym} turns out to be surprisingly simple.

Theorem 2.3. *For* $D > 0$ *we have*

$$\sigma_D^{\text{sym}} = \sum_{i=0}^{\lfloor D/2 \rfloor} \frac{(D-1)_{2i}}{i!} (D - 2i).$$

The resulting formula for $\hat{\sigma}_D$ is originally due to V. Liskovets [13]. See [17, Section 4] for discussion of symmetric LCDs.

Proof of Theorem 2.3. We are looking for the orbits of the dihedral group

$$D_{2D} = \langle \omega_D, z_D \rangle \subset S_{2D},$$

where $\omega_D(i) = 2D + 1 - i,\ 1 \leqslant i \leqslant 2D$. We have

$$\#D_{2D} = \begin{cases} 4D, & D > 1, \\ 2, & D = 1. \end{cases}$$

For $D > 1$ we have by Burnside's lemma

$$\hat{\sigma}_D = \frac{1}{4D} \sum_{i=0}^{2D-1} r_{z_D^i} + r_{\omega_D \cdot z_D^i}.$$

This is however also true for $D=1$ (since we count both elements twice and divide by the double number). Then

$$\sigma_D^{\mathrm{sym}} = 2\hat{\sigma}_D - \sigma_D = \frac{1}{2D}\sum_{i=0}^{2D-1} r_{\omega_D \cdot z_D^i}$$

Lemma 2.3. *There is a bijection from $R_{\omega_D \cdot z_D^i}$ to $\Gamma_{D-i\%2,2}$.*

Using this lemma we get

$$\sigma_D^{\mathrm{sym}} = \tfrac{1}{2}\{\gamma_{D,2} + \gamma_{D-1,2}\}$$

from which the formula follows by an easy transformation. □

Proof of lemma. The bijection from $\Gamma_{D-i\%2,2}$ to $R_{\omega_D \cdot z_D^i}$ can be described as follows. Note that $\omega_D \cdot z_D^i$ acts as a transpositive involution on $\{1,\ldots,i\}$ and $\{i+1,\ldots,2D\}$, reversing the order of all elements in these sets.

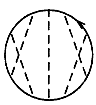
Define a map $m_i : \{1,\ldots,D-i\%2\} \to \mathscr{P}(\{1,\ldots,2D\})$ by

$$m_i(j) = \begin{cases} \{j, 1+i-j\}, & j \leqslant \lfloor \frac{i}{2} \rfloor, \\ \{\lfloor \frac{i+1}{2} \rfloor + j, 1+2D+\lfloor \frac{i}{2} \rfloor - j\}, & D-i\%2 \geqslant j > \lfloor \frac{i}{2} \rfloor. \end{cases}$$

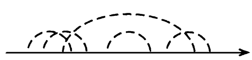
For a self-loop at basepoint j make a chord between the two points in $m_i(j)$. For a chord between basepoints j_1 and j_2 connect the four basepoints of $m_i(\{j_1,j_2\})$ in two pairs by connecting $\min(m_i(j_1))$ with $\min(m_i(j_2))$ for a chord labeled by 1 and to $\max(m_i(j_2))$ if the label is 2. Finally, if i is odd, connect basepoints $\lfloor (i+1)/2 \rfloor$ and $\lfloor (i+1)/2 \rfloor + D$. One can check that this procedure indeed describes a bijection. □

2.5. Degenerate CDs and LCDs

Definition 2.2. Let a CD (or LCD) be *degenerate*, if it has an *isolated* chord, i.e. one not crossed by any other, e.g. like in the following diagrams [1]



or



Degenerate CDs or LCDs occur in Vassiliev theory as so-called *FI* relations, imposing such diagrams to be zero in the chord diagram algebra. So the count of *FI* relations is the same as the count of degenerate CDs or LCDs. Let

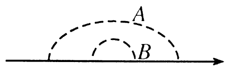
$$\omega_D = \#\{\text{degenerate CDs of deg } D\}, \quad \psi_D = \#\{\text{degenerate LCDs of deg } D\}$$

and $\bar{\omega}_D$ and $\bar{\psi}_D$ the corresponding counts of non-degenerate CDs and LCDs.

For the explanation of our approach it will be helpful to introduce the following self-suggesting notions.

[1] The cyclic one is different from the closing up of the linearized one.

Definition 2.3. We will say that a chord A of an LCD (or GLCD) *encloses* another chord (or a self-loop) B (or B is *enclosed* by A, or B is *inside* of A), if the endpoints of B are between the endpoints of A.



Conversely, we will say that B is *outside* of A if A does not enclose B (which does *not* mean that B encloses A !). A chord will be called *minimal* if it does not enclose another chord. In the same way it will be called *maximal* if there is no other chord enclosing it.

Another definition we will need later is the following one.

Definition 2.4. The *length* of a chord A in a LCD is the number of basepoints of other chords between the two basepoints of A, augmented by 1. The length of a chord in a CD will be the minimum of its lengths counted on both circle segments between its endpoints. E.g., the CD in Definition 2.2 has four chords of length 2 and one of length 5.

We will start by counting degenerate LCDs of degree D. Applying the inclusion–exclusion principle and grouping by the number of minimal isolated chords on LCDs we get a recursive formula for ψ_D.

$$\psi_D = \sum_{i=1}^{D} (-1)^{i-1} \sum_{\substack{(j_1,\dots,j_i,k) \\ j_l,k\geqslant 0 \\ j_1+\cdots+j_l+k=D-i}} \lambda_{k,i+1} \prod_{l=1}^{i} (\lambda_{j_l} - \psi_{j_l}) \tag{5}$$

and $\psi_0 = 0$. Here i is the number of choice of minimal isolated chords, $j_1,\dots,j_i$ are the degrees of the LCDs enclosed by the i chords, and k is the degree of the remaining sl-diagram.

Using the characteristic series $P_{\bar\psi}$ and P_λ defined by

$$P_{\bar\psi}(x) = \sum_{i=0}^{\infty} \bar\psi_i x^i$$

and

$$P_\lambda(x, y) = \sum_{i,l=0}^{\infty} x^i y^l \lambda_{i,l+1},$$

(5) can be rewritten more nicely as

$$P_{\bar\psi}(x) = P_\lambda(x, -x P_{\bar\psi}(x)).$$

For determining ω_D we have to work a little harder. We will calculate the number $\tilde{\gamma}_{d,c}$ of GLCD's of degree d and cycl. c, which produce[2] degenerate CDs.

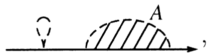
In the case $c=1$ we have

$$\tilde{\gamma}_{d,1} := \begin{cases} \psi_{d/2} & \text{if } 2 \mid d, \\ 0 & \text{else.} \end{cases}$$

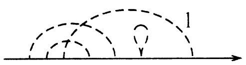
So, from now on let $c \geqslant 2$. We will distinguish two cases.

Case 1: There is an isolated chord in the CD coming from a self-loop in the GLCD.

In this case we must have $c=2$ and exactly one . If we cut the CD just before the chord coming from the self-loop, we get (applying $\Phi_{D,2}^{-1}$ of Theorem 2.1) a GLCD which looks like



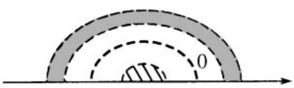
where A is a non-degenerate LCD.[3] We see that such a GLCD is of odd degree. All other GLCDs producing the same GLCD are generated by the action of $\mathbf{Z}_d$ (described in the proof of Theorem 2.1) from our special one above. They can be described as follows: Put between the basepoints of a non-degenerate LCD a and colour the chords by 1 if they enclose the self-loop and by 0 otherwise, e.g.



So for odd d there are $d \cdot \tilde{\psi}_{(d-1)/2}$ such GLCDs.

Case 2: All isolated chords in the CD come from chords in the GLCD.

Let $\eta_{c,d}$ be the number of such GLCDs. Take a minimal chord C in the GLCD producing an isolated chord. Then it must be coloured by 0 or $c-1$ (else its c copies would mutually intersect in the CD). If it has label 0 then it only encloses 0-labeled chords, which have to build up a non-degenerate LCD.



If it is labeled $c-1$, we can use the $\mathbf{Z}_d$ action to transform it into a 0-labeled chord. This way we see that outside of a $c-1$-labeled chord there are no self-loops and the chords have a unique labeling – $c-1$ if they enclose C, and 0 otherwise. Furthermore,

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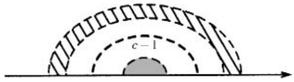
[2] From now on we will always mean this in the sense described in the proof of Theorem 2.1.

[3] Let's adopt from now on the convention in diagrams always to indicate by a gray filled part an *arbitrary* LCD and by a shaded part a *non-degenerate* LCD.

by forgetting the labels they build up a non-degenerate LCD.



We will count GLCDs of both types by the inclusion–exclusion principle over minimal chords producing isolated chords. So we have to count a GLCD with at least k such chords, so that each GLCD with exactly n chords is counted $\binom{n}{k}$ times.

There are two cases.

Case 2.1: All k chosen minimal chords are labeled by 0. Let $\xi^k_{c,d}$ be the resulting number. We can calculate it by contracting the chords and taking into account the LCDs they enclose. So we have

$$\begin{aligned}\xi^k_{c,d} &= \sum_{(e_1,\dots,e_k)\geqslant 0}\prod_{j=1}^{k}\bar{\psi}_{e_j}\cdot\lambda^c_{d-2k-2\sum e_j,k+1}\\ &= [(P_{\bar{\psi}}(x^2))^k P_{\lambda^c_{*,k+1}}(x)]_{d-2k}\quad\text{for } 0\leqslant 2k\leqslant d,\end{aligned}$$

where

$$\lambda^c_{e,d} = \binom{e+d-1}{e}\cdot\gamma_{e,c}$$

is the number of generalized string link diagrams of d strands, cyclicity c and degree e (with the obvious definition and counted by the same idea as in Lemma 2.2) and $P_{\lambda^c_{*,k}}(x)$ is the characteristic series in x of $\lambda^c_{d,k}$ over the degree[4] d

$$P_{\lambda^c_{*,k}}(x) = \sum_{d=0}^{\infty} x^d\lambda^c_{d,k}.$$

Case 2.2: There are $k-1$ chords with colour 0 and one chord with colour $c-1$. Let $\bar{\xi}^k_{c,d}$ be this number. Such a GLCD we can describe by the GLCD outside of the $c-1$-coloured chord (whose degree we will call e) with a position marked between its basepoints (where the $c-1$-coloured chord and what it encloses is attached) and by the GLCD enclosed by the chord, where $k-1$ chords of colour 0 remain, and which has to be counted as in case 2.1. So

$$\begin{aligned}\bar{\xi}^k_{c,d} &= \sum_{e\geqslant 0}(2e+1)\bar{\psi}_e\cdot\xi^{k-1}_{c,d-(2e+2)}\\ &= \left[\frac{\partial}{\partial x}\left(x\cdot P_{\bar{\psi}}(x^2)\right)\cdot P_{\xi^{k-1}_{c,*}}(x)\right]_{d-2}.\end{aligned}$$

Let

$$\zeta^k_{c,d} = \xi^k_{c,d} + \bar{\xi}^k_{c,d}. \tag{6}$$

[4] By P with a subscript containing a '$*$' we will always denote the characteristic series of the expression in the subscript over the variable replacing the '$*$'.

Now by inclusion–exclusion principle we get

$$\eta_{c,d} = \sum_{k=1}^{\lfloor d/2 \rfloor} (-1)^{k-1} \zeta_{c,d}^{k} \tag{7}$$

Putting all together, we find that

$$\tilde{\gamma}_{d,2} = \eta_{2,d} + \begin{cases} d\bar{\psi}_{(d-1)/2} & \text{if } 2d, \\ 0 & \text{else,} \end{cases}$$
$$\tilde{\gamma}_{d,c} = \eta_{c,d} \text{ for } c > 2.$$

Having $\tilde{\gamma}_{d,c}$, now we can apply Burnside's lemma and get

$$\omega_D = \frac{1}{2D} \sum_{d \cdot c = 2D} \varphi(c) \tilde{\gamma}_{d,c}. \tag{8}$$

2.6. CDs with chords of length 1

Let ω_D^1 be this number.[5] Determining it is nothing but a slight modification of what we did above.

Following the same strategy, first we compute $\bar{\psi}_D^1$.

We look at a LCD whose closure produces a CD with an isolated minimal chord of length 1. Such a LCD either has such a chord or it has a maximal chord, which is isolated.[6] Once again we apply the inclusion–exclusion principle. For fixed number k of chords we have again as in Section 2.5 two cases.

Case 1: All k chosen chords are minimal. By removing them we are left with a string link count.

Case 2: There are $k-1$ minimal chords and one maximal chord. By removing the maximal chord we get back to case 1.

So we get

$$\bar{\psi}_D^1 = \sum_{k=0}^{D} (-1)^k (\lambda_{D-k,k+1} + \lambda_{D-k,k}). \tag{9}$$

For $D = 1$ the formula gives $\psi_0^1 = -1$, since in

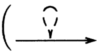
the chord is both of length 1 and maximal and is counted twice. So set $\psi_1^1 = 0$.

For the computation of $\tilde{\gamma}_{d,c}^1$ for $c \geqslant 2$ we make the same case distinction as above.

[5] We will henceforth denote the equivalents of the symbols in Section 2.5 by an additional superscript.

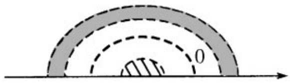
[6] i.e., not enclosed by *any* chord, not only by non-isolated chords!

Case 1: There is an isolated chord on the CD coming from a self-loop in the GLCD. This is possible in only one case $-c=2$, $d=1$ and the GLCD is

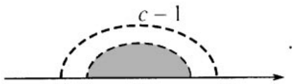
( , 2) .

Case 2: All isolated chords in the CD come from chords in the GLCD.

This count goes like in Section 2.5, but here we have no enclosed or enclosing LCDs of minimal or maximal chords. The GLCD's look like



and



.

We obtain

$$\zeta^{k,1}_{c,d} = \lambda^{c}_{d-2k,k+1} \quad \text{for } 0 \leqslant 2k \leqslant d,$$
$$\bar\zeta^{k,1}_{c,d} = \zeta^{k-1,1}_{c,d-2} = \lambda^{c}_{d-2k,k}$$

and

$$\tilde\gamma^{1}_{d,1} = \begin{cases} \bar\psi^{1}_{d/2} & \text{if } 2|d, \\ 0 & \text{else}, \end{cases}$$
$$\tilde\gamma^{1}_{d,2} = \eta^{1}_{2,d} + \begin{cases} 1 & \text{if } d=1, \\ 0 & \text{else}, \end{cases}$$
$$\tilde\gamma^{1}_{d,c} = \eta^{1}_{c,d} \quad \text{for } c>2$$

with the analogous formulas as (6) and (7) for $\zeta^{k,1}_{c,d}$ and $\eta^{1}_{c,d}$.

The formula for ω^{1}_{D} is then the same as (8).

Remark 2.1. The sequence ω^{1}_{D} (and probably also σ_D) was first calculated for $D \leqslant 9$ without a formula by direct enumeration by D. Bar-Natan [1]. It appeared in the algorithm he uses to compute the dimension of the space of weight systems (see therein table in Section 6.1, second last row).

2.7. CDs only with isolated chords

We will call such CDs also *fully degenerate* and will denote their number by ω^{2}_{D}.

This enumeration problem and the formula for it are classical. However, I include it here, because we will just see how easily it can be reproduced using our approach.

Let us start once again with the linear case.

Exercise 2.1. The number of LCDs of degree D only with isolated chords is the Catalan number

$$\psi^{2}_{D} = C_D = \frac{(2D)!}{D!\,(D+1)!}.$$

This number is the number of ways to parenthesize $D+1$ factors in a non-comm. algebra or the number of binary planar trees with a basepoint and D tops.

Hint: Group such LCDs by maximal chords and prove for the generating series P_C

$$P_C(x) = \frac{1}{1 - xP_C(x)}.$$

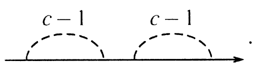
For calculating $\tilde{\gamma}^2_{d,c}$ for $c \geqslant 2$ make the following case distinction:

Case 1: There is a self-loop in the GLCD. This goes analogously to Section 2.5. We have $c=2$ and d odd, and the count is $d \cdot \psi^2_{(d-1)/2}$.

Case 2: There is no self-loop in the GLCD.

This means that d is even. By the same argument as in Section 2.5 all chords must be labeled either by 0 or by $c-1$. Furthermore, the GLCD has the following two properties:

- a chord of colour 0 encloses only chords of colour 0.
- for each 2 chords of colour $c-1$ one encloses the other, i. e. we never have something like this



Then distinguish once again two cases.

Case 2.1: All chords are coloured by 0. There are $\psi^2_{d/2}$ such GLCDs.

Case 2.2: There is a chord coloured by $c-1$.

Then by the above properties there exists a *unique* minimal chord coloured by $c-1$ and once given this the colouring of the others is uniquely determined. So such GLCDs correspond to LCDs with a distinguished chord, and their number is $d/2 \cdot \psi^2_{d/2}$.

So

$$\eta^2_{c,d} = \begin{cases} (\frac{d}{2}+1)\psi^2_{d/2} & \text{if } 2|d, \\ 0 & \text{else.} \end{cases}$$

Hence we get

$$\tilde{\gamma}^2_{d,1} = \begin{cases} \psi^2_{d/2} & \text{if } 2|d, \\ 0 & \text{else,} \end{cases}$$

$$\tilde{\gamma}^2_{d,2} = \eta^2_{2,d} + \begin{cases} d\psi^2_{(d-1)/2} & \text{if } 2d, \\ 0 & \text{else,} \end{cases}$$

$$\tilde{\gamma}^2_{d,c} = \eta^2_{c,d} \quad \text{for } c > 2$$

Table 1

D	1	2	3	4	5	6	7	8	9	10
σ_D	1	2	5	18	105	902	9 749	127 072	1 915 951	32 743 182
σ_D^{sym}	1	2	5	16	53	206	817	3 620	16 361	80 218
$\hat{\sigma}_D$	1	2	5	17	79	554	5 283	65 346	966 156	16 411 700
ω_D	1	1	3	11	70	607	6 577	85 198	1 276 563	21 695 178
ω_D^1	1	1	3	11	69	602	6 531	84 737	1 271 143	21 623 667
ω_D^2	1	1	2	3	6	14	34	95	280	854

and

$$\omega_D^2 = \frac{1}{2D} \sum_{d\cdot c=2D} \varphi(c)\tilde{\gamma}^2_{d,c}$$

$$= \frac{1}{2D}\left\{ \sum_{\substack{c\cdot d=D\\ c\geqslant 2}} \varphi(c)(d+1)C_d + C_D \right\} + \begin{cases} \frac{1}{2}C_{(D-1)/2} & \text{if } 2D, \\ 0 & \text{else,} \end{cases} \tag{10}$$

which is the classical formula for the number of planar trees with $D+1$ nodes [16].

Remark 2.2. There is a direct bijection between latter and fully degenerate CDs given by choosing a point in any bounded connected component (in the plane) of the complement of the CD and connecting points in neighbored components.

Remark 2.3. Using similar arguments it should be also possible to count the various kinds of Gauss diagrams (CDs with oriented chords) [15]. There we have to orient each chord in the GLCD and we have no self-loops.

2.8. Some computations

With the previous formulas it is not hard to compute the beginning of the various integer sequences above. [7] The first 10 values are given in Table 1.

2.9. Asymptotics

A first fact to mention is the (not very surprising) observation

Lemma 2.4.

$$\sigma_D \asymp \frac{(2D-1)!!}{2D}.$$

This is, the contribution to σ_D in (3) coming from $\gamma_{2D,1} = \lambda_D$ is the dominating one.

[7] A MATHEMATICA™ package doing this is available on my WWW page.

Proof. Using the bound

$$\gamma_{d,c} \leqslant (1+\sqrt{c})^d (d-1)!!,$$

following directly from (2), (1) and that the function $\sqrt[n]{1+\sqrt{n}}$ is monotonously falling for $n>0$ we obtain

$$\sum_{\substack{c|2D\\ c\geqslant 2}} \varphi(c)\gamma_{2D/c,c} \leqslant (1+\sqrt{2})^D (2D-1)\left(2\left\lfloor\frac{D}{2}\right\rfloor -1\right)!!$$

$$= \left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)^D \left(\frac{\sqrt{2}}{D}\right)^{D\%2} (2D-1)\frac{D!}{\lfloor\frac{D}{2}\rfloor!}.$$

Therefore,

$$\frac{\sum_{c|2D,c\geqslant 2}\varphi(c)\gamma_{2D/c,c}}{\gamma_{2D,1}} \leqslant \frac{(2+\sqrt{2})^D(2D-1)}{\binom{2D}{D}\lfloor\frac{D}{2}\rfloor!} \underset{D\to\infty}{\longrightarrow} 0. \qquad \square$$

Something more interesting happens in the case ω_D^1. Looking at (9) we see that we can write the ratio between λ_D and the kth term in the sum on the r.h.s.

$$\frac{\lambda_{D-k,k+1}+\lambda_{D-k,k}}{\lambda_D} = \frac{1}{k!}P(D),$$

where $P(D)$ is a polynomial fraction of degree 0 in D bounded above by 1 and converging to 1 for $D\to\infty$. This means that

$$\frac{\bar{\psi}_D^1}{\lambda_D} \underset{D\to\infty}{\longrightarrow} \frac{1}{e},$$

where e is the Euler number 2.71828..., and together with Lemma 2.4 we get the same for CDs.

Lemma 2.5. *Asymptotically* $1/e$ *of all CDs and LCDs have no isolated chord* (*or isolated chord of length* 1).

In fact, it is an easy exercise to convince oneself that there are 'very few' degenerate CDs with no chord of length 1, that is

$$\frac{\omega_D-\omega_D^1}{\sigma_D} \underset{D\to\infty}{\longrightarrow} 0.$$

Unfortunately, computing more carefully the difference

$$\frac{1}{e} - \frac{\bar{\psi}_D^1}{\lambda_D}$$

we see that the dominating term is

$$\frac{1}{2(2D-1)},$$

so we cannot hope for a fast convergence.

Problem. At present I do not know the asymptotics of σ_D^{sym}. (Of course, σ_D^{sym} grows much weaker than σ_D, so we have $\hat{\sigma}_D \asymp 1/2\sigma_D$.)

Remark 2.4. One can show that

$$\sigma_D^{\mathrm{sym}} = ((1+x)\mathrm{e}^{x+x^2})^{(D-1)}(0),$$

by looking at the normalized generating series of $\gamma_{*,2}$

$$\tilde{P}_{\gamma_{*,2}}(x) = \sum_{k=0}^{\infty} \frac{\gamma_{k,2}}{k!} x^k$$

and proving that $\tilde{P}_{\gamma_{*,2}}$ is a solution of the differential equation

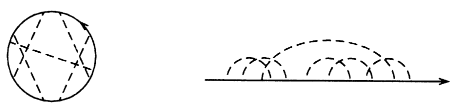
$$xf'(x) = xf(x) + 2x^2 f(x), \quad f(0) = 1.$$

3. Connected and tree-connected chord diagrams

Developing the technique introduced in Section 2 we describe the enumeration of two other kinds of chord diagrams. Although (according to one of the referees) our approach may not be the most effective one, we felt it would be interesting to incorporate both problems into our setting.

3.1. Connected CDs and LCDs

Definition 3.1. A CD (or LCD) is called *connected*, if the graph remaining after removing the solid circle line is connected.



Knot-theoretically these diagrams span the graded subspace of *primitive* Hopf algebra elements (recall [1] that chord diagrams form a Hopf algebra).

Recall the enumeration of degenerate CDs in Section 2. The present enumeration problem of

$$\omega_D^{\mathrm{nc}} = \#\{\text{non-connected CDs of deg } D\}, \quad \bar{\omega}_D^{\mathrm{nc}} = \#\{\text{connected CDs of deg } D\}$$

is very similar to this.

Start with the linear case and look for a formula for

$$\psi_D^{\mathrm{nc}} = \#\{\text{non-connected LCDs of deg } D\},$$
$$\bar{\psi}_D^{\mathrm{nc}} = \#\{\text{connected LCDs of deg } D\}.$$

Non-connected LCDs have a *virtual* separating arc (i.e., they can be extended by such an arc), which means that the arc is isolated and the enclosed and outside pieces of the LCD (with respect to this arc) are both non-empty.

Apply the inclusion–exclusion principle over the *minimal* virtual separating arcs (i.e., such ones which enclose *connected* LCDs). Note, that such arcs lie beside each other.

The recursive formula we obtain for $\bar{\psi}^{\mathrm{nc}}_D$ is

$$\bar{\psi}^{\mathrm{nc}}_0 = 1$$

and

$$\bar{\psi}^{\mathrm{nc}}_D = \sum_{i=0}^{D} (-1)^i \sum_{\substack{j_1,\dots,j_i,k \\ j_i>0 \\ k\geqslant 0,k>0 \text{ for } i=1 \\ \sum j_i+k=D}} \lambda_{k,i+1} \prod_{l=1}^{i} \bar{\psi}^{\mathrm{nc}}_{j_l}$$

$$= \sum_{i=0}^{D} (-1)^i \left[(P_{\bar{\psi}^{\mathrm{nc}}}(x) - 1)^i \cdot \left(P_{\lambda_{*,i+1}}(x) - \begin{Bmatrix} 1 \text{ if } i=1 \\ 0 \text{ else} \end{Bmatrix} \right) \right]_{x^D} \quad \text{for } D>0,$$

using the characteristic series

$$P_{\bar{\psi}^{\mathrm{nc}}}(x) = \sum_{i=0}^{\infty} \bar{\psi}^{\mathrm{nc}}_i x^i.$$

Here i is the number of choice of minimal virtual separating chords, $j_1,\dots,j_i$ are the degrees of the LCD's enclosed by the i chords, and k is the degree of the remaining sl-diagram. (Recall Section 2, that sl-diagrams arise from LCDs by cutting the solid line.)

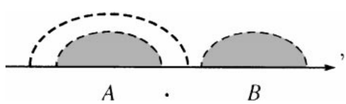
Note, that the condition of the non-emptyness of the enclosing sl-diagram is necessary only in the case of 1 virtual separating chord.

The number of non-connected LCDs is then

$$\psi^{\mathrm{nc}}_D = \lambda_D - \bar{\psi}^{\mathrm{nc}}_D.$$

Now look at the number $\tilde{\gamma}^{\mathrm{nc}}_{d,c}$ of GLCDs of degree d and cycl. $c\geqslant 2$, producing (in the sense described in Section 2) non-connected CDs.

Look at the GLCD producing such a CD obtained by cutting the solid line just before one of the endpoints of such an arc.[8] Such a GLCD can be incorporated into the following pattern (where the arc drawn is the virtual one):



where B is arbitrary and A has only 0-colored chords.[9]

[8] 'Before' we mean with respect to the orientation of the solid line.

[9] This is clear in the case where the length of the separating arc is shorter than the degree of the GLCD. In the other case note that the separating arc intersects the one from the next cycle, and smoothing out this intersection we obtain a new separating arc whose length is exactly the degree of the GLCD, so we ge back to the picture above with B equal to the empty GLCD.

So we need to enumerate such GLCDs as above and the ones resulting from them by the cyclic group action described in Section 2.

Let us apply as in the case of degenerate CDs the inclusion–exclusion principle on minimal virtual separating chords in the GLCD. As in the case in Section 2, minimal virtual separating chords can have two *virtual* colorings-0 (as in the above picture) and $c-1$ (if the action of the cyclic group has reversed the endpoint order; in this case such a chord contains outside of itself a connected GLCD with chords colored by 0 if both endpoints are on the same side of the minimal virtual separating chord and by $c-1$ otherwise).

Let us recompute the numbers $\xi^k_{c,d}$ and $\bar{\xi}^k_{c,d}$ of Section 2 for our new case. The formulas now look this way:

$$\begin{aligned}\xi^k_{c,d} &= \sum_{(e_1,\ldots,e_k)>0}\prod_{j=1}^{k}\bar{\psi}^{\mathrm{nc}}_{e_j}\cdot\lambda^c_{d-2\sum e_j,k+1}\\ &= [(P_{\bar{\psi}^{\mathrm{nc}}}(x^2)-1)^k P_{\lambda^c_{*,k+1}}(x)]_{x^d}\quad\text{for } 0\leqslant k\leqslant d,\\ \xi^k_{c,d} &= 0\quad\text{for } k>d\quad\text{or } k<0\end{aligned}\tag{11}$$

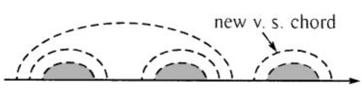
and with

$$P_{\xi^k_{c,*}}(x)=\sum_{d=0}^{\infty}\xi^k_{c,d}x^d,$$

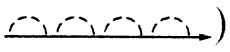
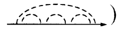
$$\begin{aligned}\bar{\xi}^k_{c,d} &= \sum_{e>0}(2e-1)\bar{\psi}^{\mathrm{nc}}_e\cdot\xi^{k-1}_{c,d-2e}\\ &= \left[\left(\frac{\partial}{\partial x}(x\cdot P_{\bar{\psi}^{\mathrm{nc}}}(x^2))-2P_{\bar{\psi}^{\mathrm{nc}}}(x^2)+1\right)\cdot P_{\xi^{k-1}_{c,*}}(x)\right]_{x^d}.\end{aligned}\tag{12}$$

The main differences to the previous case are

1. LCDs enclosed by virtual separating chords are *non-empty*, so $e_j>0$ in (11) and analogously $e>0$ in (12).
2. We get an index translation for the sl-diagram in (11), since the chords are *virtual*, i.e. they do not really belong to the GLCD.
3. In the case of $\bar{\xi}^k_{c,d}$ we must replace the factor '$(2e+1)$' in Section 2 by '$(2e-1)$'. The reason is the following: If we allow a minimal virtual separating chord of color $c-1$ to contain all basepoints of the LCD outside of it on *one* side only, then we can also draw a minimal virtual separating chord of color 0 around this component.



This way we obtain diagrams with positions of minimal virtual separating chords which are not incorporated in either counts $\xi^k_{c,d}$ (which counts pictures like

) and $\bar{\xi}^k_{c,d}$ (which counts pictures like ). We can avoid this by declaring that we draw a virtual separating chord of color $c-1$ *only* if the component outside of this chord would *enclose* it, i.e., has basepoints on *both* sides of the virtual separating chord (else we draw a virtual separating chord of color 0 around the component).

4. Note, that the non-emptyness of the enclosing sl-diagram is not required here.

The rest of the formulas then remain the same as for degenerate CDs.

$$\zeta^k_{c,d} = \xi^k_{c,d} + \bar{\xi}^k_{c,d},$$

$$\eta_{c,d} = \sum_{k=1}^{d} (-1)^{k-1} \zeta^k_{c,d}.$$

Then we have

$$\tilde{\gamma}^{\mathrm{nc}}_{d,1} = \psi^{\mathrm{nc}}_{d/2} \quad \text{for } 2|d,$$

$$\tilde{\gamma}^{\mathrm{nc}}_{d,1} = 0 \quad \text{for } 2\nmid d,$$

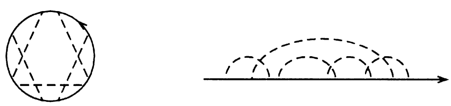
$$\tilde{\gamma}^{\mathrm{nc}}_{d,c} = \eta_{c,d} \quad \text{for } c \geqslant 2$$

and finally

$$\omega^{\mathrm{nc}}_D = \frac{1}{2D} \cdot \sum_{c|2D} \varphi(c) \cdot \tilde{\gamma}^{\mathrm{nc}}_{2D/c,c}.$$

3.2. Tree-connected CDs and LCDs

Definition 3.2. A CD (or LCD) of degree D is called *tree-connected*, if it is connected and there are exactly $D-1$ chord intersections, i.e., the intersection graph of the dashed chords is a tree.



This definition appears in a somewhat different form in [14, Section 3] and [6, Section 2]. See [4] for a different application of the labelled intersection graph of a LCD.

Remark 3.1. Do not confuse tree-connected CDs with tree-*like* CDs (which are the ones with no chord intersections, i.e., only with isolated chords, and are called in Section 2 'fully degenerate').

Let

$$\omega^{\mathrm{tc}}_D = \#\{\text{tree-connected CDs of deg } D\}.$$

Here we have a little bit more differences to the counts in Section 2.

Start with the linear case. To enumerate tree-conn. LCDs

$$\bar{\psi}^{\mathrm{tc}}_D = \#\{\text{tree-connected LCDs of deg } D\},$$

use the following idea.

Start with the left-most chord of a tree-connected LCD and walk along the chord (in the direction given by the orientation of the solid line) until we intersect a second chord, then walk (in the same direction) along this chord until we get to a third chord, and repeat this way until we walk to the end of the last chord and land back on the solid line. The chords (segments of which) we have passed, form a sub-LCD C which looks like this

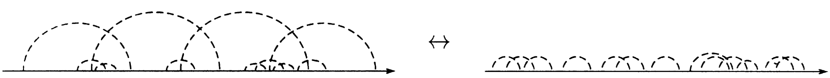
(13)

Let us denote for a moment by d its degree. All the remaining chords are attached at the $2d-2$ inner basepoints of this sub-LCD in such a way that we obtain a (non-empty) tree-connected LCD of lower degree for the fixed basepoint P by applying the following procedure:

Case 1: If this basepoint is a right basepoint remove all chords which are not enclosed by the right neighbor of the chord ending on P in (13) except the chord ending on P itself.

Case 2: If this basepoint is a left basepoint remove all chords which are not enclosed by the left neighbor of the chord ending on P in (13) except the chord ending on P itself and put the rightmost basepoint to the left.

Here is an example:



Conversely, from any collection of $2d-2$ non-empty tree-conn. LCDs we can reconstruct a tree-conn. LCD by inverting this procedure. The recursive formula for $\bar{\psi}^{\mathrm{tc}}_D$ is then immediate:

$$\bar{\psi}^{\mathrm{tc}}_0 = 1,$$

$$\bar{\psi}^{\mathrm{tc}}_D = \sum_{i=1}^{D} [(P_{\bar{\psi}^{\mathrm{tc}}}(x) - 1)^{2i-2}]_{x^{D+i-2}} \quad \text{for } D > 0 \tag{14}$$

with

$$P_{\bar{\psi}^{\mathrm{tc}}}(x) = \sum_{i=0}^{\infty} \bar{\psi}^{\mathrm{tc}}_i x^i.$$

(The sum over $k=1$ makes sense to get from $D=0$ to 1).

Remark 3.2. Leroux and Miloudi [12, (5.16)] and also Dulucq and Penaud [9, Theorem 2.2] found the explicite formula

$$\bar{\psi}^{\mathrm{tc}}_D = \frac{1}{D-1}\binom{3D-3}{D-2}, \quad D \geqslant 2.$$

In fact, one can derive from (14) that

$$g(x) = \frac{P_{\bar{\psi}^{\mathrm{tc}}}(x) - 1}{x}$$

is the solution of Eq. (5.15) of [12]

$$g(x) = 1 + xg^3(x).$$

Let us now consider cyclicity $c \geqslant 2$ and count GLCDs producing tree-connected CDs. Let us adopt the convention that we separate the solid line of a CD into $2D$ *equally* long arcs and draw chords always as *straight* lines.

First look at the tree-connected CD (drawn this way) coming from the GLCD. Temporarily remove all chords connecting opposite basepoints (i.e., *self-loop chords*). Then the middlepoint of the circle belongs to a component of the complement of the CD.

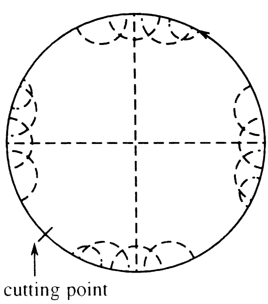
If this component is not bounded by any piece of the solid line then we have an intersection loop of chords and our CD is not tree-connected.

If there is such a piece, then there are at least c ones and the CD has $\geqslant c$ components. In order the CD to be connected we must connect all components by reinstalling the self-loop chords. So we need at least 1 self-loop chord. On the other hand we must have < 3 self-loop chords (since three self-loop chords produce an intersection cycle among themselves).

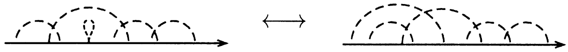
We have two cases.

Case 1: 1 . Cyclicity must be 2 (1 self-loop chord) or 4 (2 self-loop chords). Degree must be *odd*.

In this case look at the GLCD produced by cutting the CD at a position which we choose to lie on a piece of the solid line bounding the component of the circle center created after removing the self-loop chords (see example).



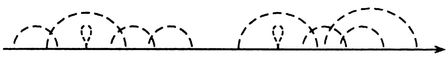
There is a bijection between such a GLCD and a tree-connected LCD by replacing the self-loop by a chord whose left end becomes the leftmost basepoint and whose right end replaces the previous position of the .



The orbits of the cyclic group action on these GLCDs are free (e.g., since the appears always at different positions if we enumerate all basepoints from left to right), i.e., we get $d\bar{\psi}^{\mathrm{tc}}_{(d+1)/2}$ GLCDs.

Case 2: 2. Cyclicity must be 2. Degree is *even*.

By the procedure of case 1 we obtain a GLCD looking like this:



(15)

If we transform the left into a chord ending on the left-most position and the right into a chord ending on the right-most position, we get a bijection to diagrams obtained by multiplication of two tree-connected LCDs. E. g., the corresponding diagram to (15) is

The count of such diagrams is

$$\sum_{\substack{e_1+e_2=d,\\ 2\nmid e_1,e_2}} \bar{\psi}^{\mathrm{tc}}_{(e_1+1)/2}\bar{\psi}^{\mathrm{tc}}_{(e_2+1)/2}.$$

Now let the cycl. group $\mathbb{Z}_d$ act on such diagrams and note that each GLCD obtained in this way is obtained always in exactly 2 ways.

- If first and second component in (15) are equal, the action of $\mathbb{Z}_d$ has period $d/2$.
- If first and second component in (15) are distinct, $\mathbb{Z}_d$ acts freely, but the orbit is equal to the one of the diagram obtained from (15) by swapping the components.

In full completeness, we obtain the following values for $\tilde{\gamma}^{\mathrm{tc}}_{d,c}$ of Section 2 in our case:

$$\tilde{\gamma}^{\mathrm{tc}}_{d,1} = \bar{\psi}^{\mathrm{tc}}_{d/2} \quad \text{for } 2|d,$$
$$\tilde{\gamma}^{\mathrm{tc}}_{d,1} = 0 \quad \text{for } 2\nmid d,$$

$$\begin{aligned}\tilde{\gamma}^{\mathrm{tc}}_{d,2} &= \frac{d}{2}\cdot\sum_{e=1}^{d/2}\bar{\psi}^{\mathrm{tc}}_{e}\bar{\psi}^{\mathrm{tc}}_{d/2+1-e}\\ &= \frac{1}{2}\cdot\left[x\cdot\frac{\partial}{\partial x}\left(\frac{1}{x^2}\cdot(P_{\bar{\psi}^{\mathrm{tc}}}(x^2)-1)^2\right)\right]_{x^d} \quad \text{for } 2|d,\end{aligned}$$

$$\tilde{\gamma}^{\mathrm{tc}}_{d,2} = d\cdot\bar{\psi}^{\mathrm{tc}}_{(d+1)/2} \quad \text{for } 2\nmid d,$$
$$\tilde{\gamma}^{\mathrm{tc}}_{d,4} = d\cdot\bar{\psi}^{\mathrm{tc}}_{(d+1)/2} \quad \text{for } 2\nmid d,$$

$$\tilde{\gamma}^{\rm tc}_{d,4} = 0 \quad \text{for } 2|d,$$
$$\tilde{\gamma}^{\rm tc}_{d,c} = 0 \quad \text{else.}$$

Finally, as always by Burnside lemma,

$$\omega^{\rm tc}_D = \frac{1}{2D} \cdot \sum_{c|2D} \varphi(c) \cdot \tilde{\gamma}^{\rm tc}_{2D/c,c}.$$

3.3. Some computations

With the previous formulas one can compute the beginning of the various integer sequences above. [10] Here are the first 10 values of each one.

D	1	2	3	4	5	6	7	8	9	10
$\omega^{\rm tc}_D$	1	1	1	2	7	25	108	492	2431	12 371
$\omega^{\rm nc}_D$	0	1	3	12	74	647	6961	89 739	1 337 152	22 609 111
$\bar{\omega}^{\rm nc}_D$	1	1	2	6	31	255	2788	37 333	578 799	10 134 071

Remark 3.3. By similar arguments as in Section 2 one can show that

$$\frac{\bar{\omega}^{\rm nc}_D}{\sigma_D} \underset{D\to\infty}{\longrightarrow} \frac{1}{e}.$$

This was first proven for the linear case by Stein and Everett [18], see also [17, p. 362].

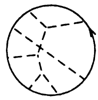
Remark 3.4. The sequence of tree-connected CDs was previously known. According to Sloane [16] it appeared in an enumeration of planar alcohol molecules [12] (which correspond to tree-connected CDs by putting on each chord crossing a ‘C’ and on each chord basepoint a ‘H’ atom and forgetting the solid line). Leroux and Miloudi used some more general and more elegant Pólya theory arguments. Our approach is however geometrically more understandable.

4. Odds & Ends

As a further challenge, it would be also interesting to know how to count the s. c. Chinese character diagrams (CCDs), which are more complicated relatives of the CDs and look something like this:

[10] A MATHEMATICA™ package with all formulas is available on my WWW page.

(formally a trivalent graph with distinguished Hamiltonian cycle, the solid line, as explained in [1]), but yet I do not know how to find a generalization of the algorithm to do this.



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